

# A COMPUTATION OF $H^1(\Gamma, H_1(\Sigma))$

RASMUS VILLEMOES

**ABSTRACT.** Let  $\Sigma = \Sigma_{g,1}$  be a compact surface of genus  $g \geq 3$  with one boundary component,  $\Gamma$  its mapping class group and  $M = H_1(\Sigma, \mathbb{Z})$  the first integral homology of  $\Sigma$ . Using that  $\Gamma$  is generated by the Dehn twists in a collection of  $2g + 1$  simple closed curves (Humphries' generators) and simple relations between these twists, we prove that  $H^1(\Gamma, M)$  is either trivial or isomorphic to  $\mathbb{Z}$ . Using Wajnryb's presentation for  $\Gamma$  in terms of the Humphries generators we can show that it is not trivial.

## 1. GROUP COHOMOLOGY IN 45 SECONDS

For  $G$  a group and  $M$  a (left)  $G$ -module (a module over the group ring  $\mathbb{Z}G$ ), a cocycle is a map  $u: G \rightarrow M$  satisfying the *cocycle condition*,

$$(1) \quad u(gh) = u(g) + gu(h),$$

for all  $g, h \in G$ . The set of  $M$ -valued cocycles on  $G$  is denoted  $Z^1(G, M)$ . A coboundary is a cocycle of the form  $g \mapsto m - gm$  for some  $m \in M$ ; the set of these is denoted  $B^1(G, M)$ . The cohomology group  $H^1(G, M)$  is the quotient  $Z^1(G, M)/B^1(G, M)$ .

It follows immediately from (1) that  $u(e) = 0$  for  $e$  the identity element of  $G$ . It also follows that  $u(g^{-1}) = -g^{-1}u(g)$ , and that

$$(2) \quad u(ghg^{-1}) = (1 - ghg^{-1})u(g) + gu(h).$$

We also note that  $u$  is determined by its values on a set of generators of  $G$ . Indeed, if  $G = \langle g_1, \dots, g_r \mid r_1, \dots, r_s \rangle$  is a finite presentation of  $G$ , the space of cocycles  $Z^1(G, M)$  may be identified with the subspace of  $M^r$  determined by the  $s$  linear equations in the  $r$  unknowns  $m_1 = u(g_1), \dots, m_r = u(g_r)$  given by the relations  $r_j$  obtained by expanding via the cocycle condition. For example, the  $(\mathbb{Z}G)$ -linear equation associated to the relation  $g_1 g_2 g_3^{-1} g_1 = e$  is

$$(1 + g_1 g_2 g_3^{-1})m_1 + g_1 m_2 - g_1 g_2 g_3^{-1} m_3 = 0.$$

In the same setting,  $B^1(G, M)$  may be identified with the subspace

$$\left\{ \left( (1 - g_1)m, \dots, (1 - g_r)m \right) \mid m \in M \right\} \subseteq M^r.$$

If the action of  $G$  on  $M$  is trivial, a cocycle is simply a group homomorphism  $G \rightarrow M$ , and since any coboundary vanishes in this case, we have

$$H^1(G, M) = \text{Hom}(G, M) = \text{Hom}(G_{\text{ab}}, M)$$

where  $G_{\text{ab}}$  denote the abelianization of  $G$ .

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**1.1. Exact sequences.** Suppose  $1 \rightarrow K \rightarrow G \rightarrow Q \rightarrow 1$  is a short exact sequence of groups and  $M$  a  $G$ -module. The action of  $G$  on  $K$  by conjugation induces an action on the cohomology group  $H^1(K, M)$ . There is an exact sequence

$$(3) \quad 0 \rightarrow H^1(Q, M^K) \rightarrow H^1(G, M) \rightarrow H^1(K, M)^G,$$

where  $M^K$  denote the subspace of  $M$  invariant under  $K$ , and  $H^1(K, M)^G$  is the subspace invariant under the above-mentioned action of  $G$ . This is a consequence of the Hochschild-Serre spectral sequence, but can also be checked by direct verification using the hands-on definitions of cocycles and coboundaries given above.

## 2. NOTATION AND CONVENTIONS

We consider a compact, oriented surface  $\Sigma$  of genus  $g \geq 1$  with one boundary component.

**2.1. Curves and homology.** Fix a collection  $\mathcal{C}$  of  $3g - 1$  simple, closed curves  $\alpha_j, \beta_j, \gamma_j$  as shown in Figure 1. With appropriate choices of orientations (which we also fix), the homology classes  $a_j = [\alpha_j]$ ,  $b_j = [\beta_j]$ ,  $c_j = [\gamma_j]$  satisfy

$$(4a) \quad \omega(a_j, a_k) = \omega(b_j, b_k) = 0$$

$$(4b) \quad \omega(a_j, b_k) = \delta_{jk}$$

$$(4c) \quad c_j = a_{j-1} - a_j$$

where  $\omega$  denotes the intersection pairing on  $M = H_1(\Sigma, \mathbb{Z})$ . In particular,  $S = (a_1, b_1, \dots, a_g, b_g)$  is a symplectic basis for  $M$ . We let  $\mathcal{S}$  denote the subset of  $\mathcal{C}$  consisting of the  $2g$  curves  $\alpha_j, \beta_j$ . There are involutions on the sets  $\mathcal{S}$  and  $\mathcal{C}$  given by  $\alpha_j \leftrightarrow \beta_j$  and  $a_j \leftrightarrow b_j$ , respectively; we will use  $\iota$  to stand for either of these involutions. Clearly  $\iota[\eta] = [\iota\eta]$  for any  $\eta \in \mathcal{C}$ .

We will use the notation  $M_j$  for the symplectic subspace  $\text{span}_{\mathbb{Z}}(a_j, b_j)$  of  $M$ , and  $M'_j$  for its complement  $\text{span}(S - \{a_j, b_j\})$ . Associated to these are the projections  $\pi_j$  and  $\pi'_j = \text{id} - \pi_j$ .

Using  $\gamma_1$  and  $c_1$  as synonyms for  $\alpha_1$  and  $a_1$ , respectively, we observe that  $S' = (c_1, b_1, c_2, b_2, \dots, c_g, b_g)$  also constitutes a basis for  $M$ ; this is immediate from (4c). We use  $\mathcal{S}'$  to denote the set  $\{\gamma_1, \beta_1, \dots, \gamma_g, \beta_g\} \subset \mathcal{C}$ .

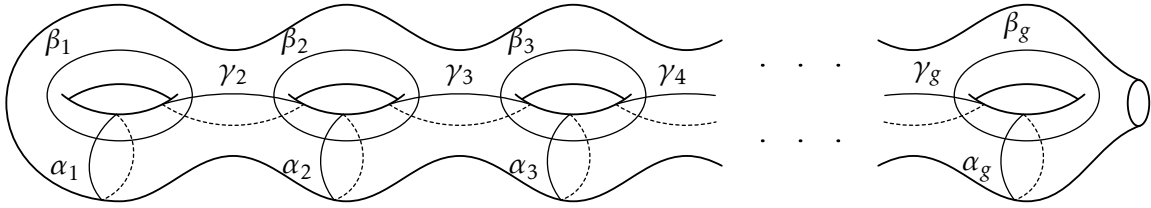


FIGURE 1. A collection of simple closed curves on  $\Sigma$ .

**2.2. Twists and action.** It is well-known that the action of the (left) Dehn twist in a simple closed curve  $\eta$  on a homology element  $m$  is given by

$$(5) \quad \tau_\eta m = m + \omega(m, [\vec{\eta}])[\vec{\eta}],$$

where  $\vec{\eta}$  denotes any of the oriented versions of  $\eta$  (see e.g. [2]). For convenience, we record these consequences:

$$\begin{aligned} (6a) \quad & \tau_{\alpha_j} a_k = a_k & \tau_{\beta_j} b_k = b_k \\ (6b) \quad & \tau_{\alpha_j} b_k = b_k & \tau_{\beta_j} a_k = a_k \\ (6c) \quad & \tau_{\alpha_j} b_j = b_j - a_j & \tau_{\beta_j} a_j = a_j + b_j \end{aligned}$$

Here, (6a) holds for any  $1 \leq j, k \leq g$ , while (6b) holds for  $j \neq k$ . For  $1 < j \leq g$  we also have

$$\begin{aligned} (7a) \quad & \tau_{\gamma_j} b_j = b_j + a_{j-1} - a_j = b_j + c_j & \tau_{\gamma_j} b_{j-1} = b_{j-1} - a_{j-1} + a_j = b_{j-1} - c_j \\ (7b) \quad & \tau_{\gamma_j} a_k = a_k \\ (7c) \quad & \tau_{\gamma_j} b_k = b_k, \end{aligned}$$

with (7b) holding for any  $k$ , and (7c) for  $k \neq j-1, j$ .

**2.3. Wajnryb's presentation for the mapping class group.** The curves in  $\mathcal{C}$  are not completely arbitrary; the twists in these curves are the so-called Lickorish generators for the mapping class group. Humphries [3] showed that the  $2g+1$  twists in  $\mathcal{S}' \cup \{\alpha_2\}$  actually suffice (and that  $2g+1$  is the minimal number of twists needed to generate the mapping class group). Later, Wajnryb [7] was able to give a finite presentation of the mapping class group using Humphries generators. An exposition of this, along with more details on the history of generating and presenting the mapping class group, can be found in [2]. We give a slightly modified version of their Theorem 5.3.

**Theorem 2.1.** *The mapping class group  $\Gamma_{g,1}$  has a presentation with a generator  $g_\eta$  for each curve  $\eta \in \mathcal{S} \cup \{\alpha_2\} = \{\gamma_1, \beta_1, \gamma_2, \beta_2, \dots, \gamma_g, \beta_g, \alpha_2\}$ , and relations:*

- (a) *If  $\eta$  and  $\lambda$  are disjoint,  $g_\eta$  and  $g_\lambda$  commute.*
- (b) *If  $\eta$  and  $\lambda$  intersect in exactly one point,  $g_\eta g_\lambda g_\eta = g_\lambda g_\eta g_\lambda$ .*
- (c) *Let  $w$  denote the word  $g_{\beta_2} g_{\gamma_2} g_{\beta_1} g_{\gamma_1} g_{\beta_1} g_{\gamma_2} g_{\beta_2}$ . Then*

$$(8) \quad (g_{\gamma_1} g_{\beta_1} g_{\gamma_2})^4 = g_{\alpha_2} w g_{\alpha_2} w^{-1}.$$

- (d) *Let*

$$\begin{aligned} w_1 &= g_{\beta_2} g_{\gamma_3} g_{\gamma_2} g_{\beta_2} & x_1 &= w_1^{-1} g_{\alpha_2} w_1 \\ w_2 &= g_{\beta_1} g_{\gamma_2} g_{\gamma_1} g_{\beta_1} & x_2 &= w_2^{-1} x_1 w_2 \\ w_3 &= g_{\beta_3} g_{\gamma_3} & x_3 &= w_3^{-1} x_1 w_3 \end{aligned}$$

and

$$w_4 = g_{\beta_3} g_{\gamma_3} g_{\beta_2} g_{\gamma_2} g_{\beta_1} x_3 g_{\gamma_1}^{-1} g_{\beta_1}^{-1} g_{\gamma_2}^{-1} g_{\beta_2}^{-1} \quad x_4 = w_4 g_{\alpha_2} w_4^{-1}.$$

Then

$$(9) \quad g_{\alpha_2} x_2 x_1 = g_{\gamma_1} g_{\gamma_2} g_{\gamma_3} x_4.$$

Of course, the abstract generator  $g_\eta$  simply corresponds to the twist  $\tau_\eta$ .

### 3. COMPUTING COHOMOLOGY

In this section we compute  $H^1(\Gamma, M)$ . We will do this by proving that any cocycle is cohomologous to one with some very nice properties. This will be done in two steps. In the first, we adapt the cocycle to the basis  $S$  (or rather, to the set of curves  $\mathcal{S}$ ), and in the second, we add another coboundary to adapt the cocycle to the basis  $S'$  for  $M$  (again, this is with respect to the set  $S'$ ). We do not specify what it means to »adapt a cocycle to a basis« or »adapt a cocycle to a set of curves«; it will be apparent from the statements of Proposition 3.1 and Proposition 3.7.

#### 3.1. Adapting to $\mathcal{S}$ .

**Proposition 3.1.** *Any cohomology class in  $H^1(\Gamma, M)$  is represented by a cocycle  $u$  satisfying that for each  $\eta \in \mathcal{S}$ , the coefficient of  $[\eta]$  in the expansion of  $u(\tau_\eta)$  in terms of the basis  $S$  is 0. Moreover, this determines  $u$  uniquely.*

*Proof.* Recall that  $\mathcal{S}$  is the collection of the  $2g$  curves  $\alpha_j, \beta_j$ , the homology classes of which constitute the basis  $S$ . We see from equations (6a), (6b) and (6c) that  $1 - \tau_{\alpha_j}$  kills all basis elements except  $b_j$ , and that  $(1 - \tau_{\alpha_j})b_j = a_j$ . Similarly,  $1 - \tau_{\beta_j}$  kills all basis elements except  $a_j$ , and  $(1 - \tau_{\beta_j})a_j = -b_j$ . If  $u$  is any cocycle, let  $x_j$  denote the coefficient of  $a_j$  in  $u(\tau_{\alpha_j})$  and  $y_j$  the coefficient of  $b_j$  in  $u(\tau_{\beta_j})$ . Adding the coboundary of the homology element

$$\sum_{j=1}^g y_j a_j - x_j b_j$$

to  $u$  produces a cocycle with the required properties. It is clear that if  $m$  is any non-zero homology element, there is some  $\eta \in \mathcal{S}$  such that  $(1 - \tau_\eta)m$  contains a non-trivial  $[\eta]$ -component, proving the uniqueness claim.  $\square$

In a sense, with this proposition we have used up all the freedom there is in the choice of cocycle representing a given cohomology class.

**Proposition 3.2.** *The cohomology group  $H^1(\Gamma, M)$  is a free abelian group of finite rank.*

*Proof.* We have just proved that there is a section  $H^1(\Gamma, M) \rightarrow Z^1(\Gamma, M)$ , which proves that there is no torsion. The claim about the finite rank follows from the fact that  $\Gamma$  is finitely generated.  $\square$

From now on, we will assume that  $u$  is a cocycle which is adapted to  $\mathcal{S}$  in the sense of Proposition 3.1. We will now proceed to determine other facts about  $u$ , using simple relations between the twists in the curves from  $\mathcal{C}$ .

First, consider the braid relation  $\tau_{\alpha_j} \tau_{\beta_j} \tau_{\alpha_j} = \tau_{\beta_j} \tau_{\alpha_j} \tau_{\beta_j}$ . Applying the cocycle condition, we obtain

$$(10) \quad u(\tau_{\alpha_j}) + \tau_{\alpha_j} u(\tau_{\beta_j}) + \tau_{\alpha_j} \tau_{\beta_j} u(\tau_{\alpha_j}) = u(\tau_{\beta_j}) + \tau_{\beta_j} u(\tau_{\alpha_j}) + \tau_{\beta_j} \tau_{\alpha_j} u(\tau_{\beta_j}).$$

Since  $u(\tau_{\beta_j})$  is a linear combination of elements from  $S - \{b_j\}$ , we have  $\tau_{\alpha_j} u(\tau_{\beta_j}) = u(\tau_{\beta_j})$ , and symmetrically,  $\tau_{\beta_j} u(\tau_{\alpha_j}) = u(\tau_{\alpha_j})$ . Hence two terms on either side of (10) cancel, and we are left with

$$(11) \quad \tau_{\alpha_j} u(\tau_{\alpha_j}) = \tau_{\beta_j} u(\tau_{\beta_j}).$$

By the normalization assumption, we have  $\pi_j u(\tau_{\alpha_j}) = y_j b_j$  and  $\pi_j u(\tau_{\beta_j}) = z_j a_j$  for some integers  $y_j, z_j$ . Applying the projection  $\pi_j$  to (11) and using that  $\pi_j$  commutes with the two twists, we obtain

$$-y_j a_j + y_j b_j = z_j a_j + z_j b_j$$

which implies  $y_j = z_j = 0$ . So we have

$$(12) \quad \pi_j u(\tau_{\alpha_j}) = \pi_j u(\tau_{\beta_j}) = 0$$

for all  $1 \leq j \leq g$ . This in turn implies that  $u(\tau_{\alpha_j}) = u(\tau_{\beta_j})$  (apply  $\pi'_j$  to (11) and use that it annihilates the twists).

We record this consequence of our calculations so far.

**Lemma 3.3.** *For  $g = 1$ ,  $H^1(\Gamma, M)$  is trivial.*

From now on we consider the case  $g \geq 2$ . For  $k \neq j$ ,  $\tau_{\alpha_j}$  commutes with  $\tau_{\alpha_k}$  and  $\tau_{\beta_k}$ . Applying the cocycle condition to these commutativity relations gives

$$\begin{aligned} u(\tau_{\alpha_j}) + \tau_{\alpha_j} u(\tau_{\alpha_k}) &= u(\tau_{\alpha_k}) + \tau_{\alpha_k} u(\tau_{\alpha_j}) \\ u(\tau_{\alpha_j}) + \tau_{\alpha_j} u(\tau_{\beta_k}) &= u(\tau_{\beta_k}) + \tau_{\beta_k} u(\tau_{\alpha_j}) \end{aligned}$$

and using the projection  $\pi_k$  along with (12) we see that

$$\begin{aligned} \pi_k u(\tau_{\alpha_j}) &= \tau_{\alpha_k} \pi_k u(\tau_{\alpha_j}) \\ \pi_k u(\tau_{\alpha_j}) &= \tau_{\beta_k} \pi_k u(\tau_{\alpha_j}). \end{aligned}$$

This clearly implies that  $\pi_k u(\tau_{\alpha_j}) = 0$ . Since this holds for any  $k$ , we obtain this important result.

**Proposition 3.4.** *A cocycle which is normalized in the sense of Proposition 3.1 vanishes on each of the Dehn twists  $\tau_{\alpha_j}, \tau_{\beta_j}$ .*

The next lemma may sound a bit cryptic, but the text following the proof should make its usefulness clear.

**Lemma 3.5.** *If  $\sigma$  is a simple closed curve disjoint from some  $\eta \in \mathcal{S}$ , then the coefficient of  $[\iota\eta] = \iota[\eta]$  in  $u(\tau_\sigma)$  is 0, where  $u(\tau_\sigma)$  is written in terms of the basis  $S$ .*

*Proof.* Since  $\sigma$  and  $\eta$  are disjoint, the associated twists commute. Applying the cocycle condition and the vanishing of  $u(\tau_\eta)$  this yields

$$u(\tau_\sigma) = \tau_\eta u(\tau_\sigma).$$

Since  $\tau_\eta(\iota[\eta]) = \iota[\eta] \pm [\eta]$  and  $\tau_\eta$  acts as the identity on all other basis element, this is only possible if  $u(\tau_\sigma)$  does not have a  $\iota[\eta]$ -component.  $\square$

Notice, for example, that this implies that  $u(\tau_{\gamma_j})$  is a linear combination of  $a_{j-1}$  and  $a_j$ , since  $\beta_{j-1}$  and  $\beta_j$  are the only curves from  $\mathcal{S}$  which  $\gamma_j$  intersect. In fact we have:

**Lemma 3.6.** *There are integers  $q_j$ ,  $j = 2, \dots, g$ , such that  $u(\tau_{\gamma_j}) = q_j a_{j-1} - q_j a_j = q_j c_j$ .*

The case  $j = 1$  is only omitted because  $a_0$  is not defined; we clearly have  $u(\tau_{\gamma_1}) = u(\tau_{\alpha_1}) = 0 = 0c_1$ .

*Proof.* The braid relation between  $\tau_{\beta_j}$  and  $\tau_{\gamma_j}$  yields

$$(13) \quad \tau_{\beta_j} u(\tau_{\gamma_j}) = u(\tau_{\gamma_j}) + \tau_{\gamma_j} \tau_{\beta_j} u(\tau_{\gamma_j})$$

after applying the cocycle condition and the vanishing of  $u(\tau_{\beta_j})$ . Using Lemma 3.5, we may write  $u(\tau_{\gamma_j}) = q_j a_{j-1} + p_j a_j$  for some integers  $q_j, p_j$ . Then the left-hand side of (13) is

$$q_j a_{j-1} + p_j a_j + p_j b_j$$

while the right-hand side is

$$(q_j a_{j-1} + p_j a_j) + (q_j a_{j-1} + p_j a_j + p_j b_j + p_j a_{j-1} - p_j a_j) = (2q_j + p_j) a_{j-1} + p_j a_j + p_j b_j.$$

From this it follows that  $p_j = -q_j$ , so we do indeed have  $u(\tau_{\gamma_j}) = q_j(a_{j-1} - a_j) = q_j c_j$ .  $\square$

Thus  $u$  is completely determined by the  $g-1$  integers  $q_2, \dots, q_g$ . In particular, the rank of  $H^1(\Gamma, H)$  is now bounded above by  $g-1$ .

We now adapt the cocycle to the set  $\mathcal{S}'$ :

**Proposition 3.7.** *Let  $u$  be a cocycle adapted to  $\mathcal{S}$  in the sense of Proposition 3.1. Then  $u$  is cohomologous to a cocycle, again denoted  $u$ , satisfying  $u(\tau_{\beta_j}) = u(\tau_{\gamma_j}) = 0$  for  $j = 1, \dots, g$ .*

Note that this new  $u$  may no longer vanish on the twists  $\tau_{\alpha_j}$  for  $j \geq 2$ .

*Proof.* Since  $(1 - \tau_{\beta_j})b_k = 0$  for all  $j, k$ , and since  $u$  already vanishes on  $\tau_{\beta_j}$  for all  $j$ , adding the coboundary of any homology element which is a linear combination of the  $b_j$  preserves this property.

Let  $q_j$  denote the integers such that  $u(\tau_{\gamma_j}) = q_j c_j$ . Put  $r_1 = 0$  and  $r_j = r_{j-1} + q_j$  for  $j > 1$ . Then

$$(1 - \tau_{\gamma_k}) \sum_{j=1}^g r_j b_j = r_{k-1} c_k - r_k c_k = -q_k c_k$$

so adding the coboundary of  $\sum_{j=1}^g r_j b_j$  produces a cocycle with the required properties.  $\square$

With these preparations, we can finally compute  $H^1(\Gamma, M)$ .

**Theorem 3.8.** *For  $g \geq 3$ , the cohomology group  $H^1(\Gamma, M)$  is isomorphic to  $\mathbb{Z}$ .*

*Proof.* By Theorem 2.1, the mapping class group is generated by the twists in the curves from  $\mathcal{S}' \cup \{\alpha_2\}$ . Since a cocycle which is adapted to  $\mathcal{S}'$  vanishes on each of the twists in these curves, we see that such a cocycle is completely determined by the element  $u(\tau_{\alpha_2}) \in M$ . In fact, we can say even more: Before we perform the adaptation to  $\mathcal{S}'$ , we have  $u(\tau_{\alpha_2}) = 0$ , so it follows by construction that

$$u(\tau_{\alpha_2}) = (1 - \tau_{\alpha_2}) \left( \sum_j r_j b_j \right) = r_2 a_2.$$

Hence the cocycle is determined by the single integer  $r_2 = q_2$ , and  $H^1(\Gamma, M)$  is thus either trivial or isomorphic to  $\mathbb{Z}$ .

To see that it is not trivial, we must prove that putting  $u(\tau_\eta) = 0$  for  $\eta \in \mathcal{S}'$  and  $u(\tau_{\alpha_2}) = a_2$  defines a cocycle. To do this, we need to check each of the relations in Wajnryb's presentation.

The disjointness and braid relations are only interesting if one of the curves is  $\alpha_2$ . But if  $\eta \in \mathcal{S}'$  is disjoint from  $\alpha_2$  (that is,  $\eta$  is not  $\beta_2$ ), we have

$$u(\tau_{\alpha_2} \tau_\eta) = u(\tau_{\alpha_2}) + \tau_{\alpha_2} u(\tau_\eta) = a_2 = \tau_\eta a_2 = u(\tau_\eta) + \tau_\eta u(\tau_{\alpha_2}) = u(\tau_\eta \tau_{\alpha_2}).$$

We also have

$$\begin{aligned} u(\tau_{\alpha_2} \tau_{\beta_2} \tau_{\alpha_2}) &= a_2 + \tau_{\alpha_2} \tau_{\beta_2} a_2 = a_2 + b_2 \\ u(\tau_{\beta_2} \tau_{\alpha_2} \tau_{\beta_2}) &= \tau_{\beta_2} a_2 = a_2 + b_2. \end{aligned}$$

Applying  $u$  to the left-hand side of (8) and expanding via the cocycle condition clearly gives 0. On the right-hand side, we get

$$u(\tau_{\alpha_2}) + u(w \tau_{\alpha_2} w^{-1}) = a_2 + (1 - w \tau_{\alpha_2} w^{-1})u(w) + w a_2$$

by (2). Clearly  $u(w)$  vanishes, and it is straight-forward to check that  $w a_2 = -a_2$  using equations (6) and (7).

Finally, we compute the values of  $u$  on the various auxilliary words occuring in (9).

$$\begin{aligned} u(x_1) &= u(w_1^{-1} \tau_{\alpha_2} w_1) = (1 - w_1^{-1})u(w_1^{-1}) + w_1^{-1} u(\tau_{\alpha_2}) = w_1^{-1} a_2 \\ &= \tau_{\beta_2}^{-1} \tau_{\gamma_2}^{-1} \tau_{\gamma_3}^{-1} (a_2 - b_2) = \tau_{\beta_2}^{-1} \tau_{\gamma_2}^{-1} (-b_2 + a_3) = \tau_{\beta_2}^{-1} (-b_2 + a_3 + a_1 - a_2) \\ &= a_1 - a_2 + a_3 \end{aligned}$$

$$\begin{aligned} u(x_2) &= w_2^{-1} u(x_1) = \tau_{\beta_1}^{-1} \tau_{\gamma_1}^{-1} \tau_{\gamma_2}^{-1} (a_1 - b_1 - a_2 + a_3) = \tau_{\beta_1}^{-1} \tau_{\gamma_1}^{-1} (a_3 - b_1) = \tau_{\beta_1}^{-1} (-a_1 - b_1 + a_3) \\ &= a_3 - a_1 \end{aligned}$$

$$\begin{aligned} u(x_3) &= w_3^{-1} u(x_1) = \tau_{\gamma_3}^{-1} \tau_{\beta_3}^{-1} (a_1 - a_2 + a_3) = \tau_{\gamma_3}^{-1} (a_1 - a_2 + a_3 - b_3) \\ &= a_1 - b_3 \end{aligned}$$

$$u(w_4) = \tau_{\beta_3} \tau_{\gamma_3} \tau_{\beta_2} \tau_{\gamma_2} \tau_{\beta_1} u(x_3) = b_1 - a_2 + b_2 + 2a_3 + b_3$$

$$u(x_4) = (1 - w_4 \tau_{\alpha_2} w_4^{-1})u(w_4) + w_4 u(\tau_{\alpha_2}) = 2a_3$$

Finally, the value of  $u$  on the left-hand side of (9) is

$$\begin{aligned} u(\tau_{\alpha_2} x_2 x_1) &= u(\tau_{\alpha_2}) + \tau_{\alpha_2} u(x_2) + \tau_{\alpha_2} x_2 u(x_1) = a_2 + (-a_1 + a_3) + (a_1 - a_2 + a_3) \\ &= 2a_3, \end{aligned}$$

whereas the value on the right-hand side is

$$u(\tau_{\gamma_1} \tau_{\gamma_2} \tau_{\gamma_3} x_4) = \tau_{\gamma_1} \tau_{\gamma_2} \tau_{\gamma_3} u(x_4) = 2a_3.$$

Hence the map  $\{\tau_\eta \mid \eta \in \mathcal{S}' \cup \{\alpha_2\}\} \rightarrow M$  defined by  $\tau_{\alpha_2} \mapsto a_2$  and  $\tau_\eta \mapsto 0$  for  $\eta \in \mathcal{S}'$  extends to a cocycle defined on  $\Gamma$ , and this cocycle represents a generator for the cohomology group  $H^1(\Gamma, M)$ .  $\square$

## 4. FINAL REMARKS

**4.1. Other computations.** Earle [1] constructed a cocycle  $\psi: \Gamma \rightarrow H_1(\Sigma, \mathbb{R})$  such that  $(2g-2)\psi$  has values in  $H_1(\Sigma, \mathbb{Z})$ . Later Morita [5] proved that  $H^1(\Gamma, H_1(\Sigma, \mathbb{Z})) \cong \mathbb{Z}$  using a combinatorial approach. Recently, Kuno [4] has computed Earle's cocycle in terms of Morita's. Satoh [6] has computed the first homology and cohomology of the automorphism and outer automorphism groups of a free group with coefficients in the abelianization of the free group. This list of references is of course by no means exhaustive.

**4.2. Surface with a marked point.** Instead of a surface with boundary, one could consider a closed surface  $\Sigma_*$  with a marked point (or equivalently, a closed surface minus a point). Denote the mapping class group of  $\Sigma_*$  by  $\Gamma_*$ .

**Proposition 4.1.** *The natural homomorphism  $\Gamma \rightarrow \Gamma_*$  induces an isomorphism  $H^1(\Gamma, M) \cong H^1(\Gamma_*, M)$ , where  $M = H_1(\Sigma) \cong H_1(\Sigma_*)$ .*

*Proof.* The map  $\Gamma \rightarrow \Gamma_*$  is obtained by gluing a disc with a marked point to the boundary of  $\Sigma$  and extending a representative of a mapping class  $f \in \Gamma$  by the identity. Clearly the inclusion  $\Sigma \rightarrow \Sigma_*$  is an isomorphism on  $H_1$ , and this is equivariant with respect to the homomorphism  $\Gamma \rightarrow \Gamma_*$ . We identify the two homology groups via this isomorphism.

Now,  $\Gamma \rightarrow \Gamma_*$  is surjective, and the kernel is the infinite cyclic group  $\langle \tau_\partial \rangle$  generated by the twist in the boundary of  $\Sigma$  [2, Proposition 3.19]. Note that  $\tau_\partial$  acts trivially on  $M$ , so  $M^{\langle \tau_\partial \rangle} = M$  and  $H^1(\langle \tau_\partial \rangle, M) = \text{Hom}(\langle \tau_\partial \rangle, M)$ . Applying the exact sequence (3) we obtain

$$(14) \quad 0 \rightarrow H^1(\Gamma_*, M) \rightarrow H^1(\Gamma, M) \rightarrow H^1(\langle \tau_\partial \rangle, M)^\Gamma = 0,$$

since a homomorphism from  $\langle \tau_\partial \rangle$  is simply an element of  $M$ , and 0 is the only  $\Gamma$ -invariant element of  $H_1(\Sigma)$ .  $\square$

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CENTRE FOR QUANTUM GEOMETRY OF MODULI SPACES, NY MUNKEGADE 118, BLDG. 1530, AARHUS UNIVERSITY, 8000 AARHUS C, DENMARK

Current address: Dept. of Mathematics, University of Maryland, College Park, MD 20742-4015, United States

E-mail address: math@rasmusvillemoes.dk